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THE PRESENT SITUATION IN THE PHILOSOPHY OF MATHEMATICS[1]

PHILOSOPHIC controversy is not always an organized warfare. It is hard to get our two opposing armies to face each other squarely on definite lines and to join issue until one side or the other has to yield. The situation is often like that of a civil war carried on through guerrilla raids. Each side sallies forth into the enemy's fields of experience, captures whatever it can carry away; and if any stubborn fact shows the least resistance, it is ruthlessly put to fire and sword. To help bring about a condition of more orderly and scientific warfare, a series of new military charts might be useful.

The following is offered primarily to outline the military geography of a distant province of the philosophic kingdom—the field known as the philosophy of mathematics; and if the author occasionally puts himself on the firing-lines, it must be remembered that military charts are not prepared by neutral civilians.

I. WHAT IS PURE MATHEMATICS?

Until very recently philosophers generally assumed it to be the science of quantity or magnitude; and magnitude was defined as that of which we can predicate the relations of equality, greater than, or less than (*cf.* Grassmann's "Ausdehnungslehre von 1844," p. 7). But the rise of distinctly non-quantitative mathematical sciences, like the theory of groups, projective geometry, or analysis situs, has rendered the necessity for a new definition imperative. One of the first, and to mathematicians most satisfactory, attempts to meet this need is that of Papperitz in the first volume of the *Jahresberichte* of the Deutsche Mathematiker Vereinigung. "The subject-matter of pure mathematics," he says, "consists of the *relations* that can be established between any objects as contained in an *ordered manifold*" (p. 36) (italics mine). Excepting, however, Professor Royce, who has on various occasions endeavored to bring out the general significance of a similar definition by Mr. Kempe,[2] this mode of defining mathematics does not seem to have received any philosophic attention. There is, however, another way of defining mathematics, and that is as a certain method of knowledge. This seems to be more inviting to philosophers with a limited knowledge of mathematics. For apparently one has only to get hold of the first piece of available mathematical knowledge, and, by subjecting it to a careful logical analysis, arrive at the essence of mathematical method. Accordingly, theories as to the nature of mathematical knowledge are too numerous to be the subject of a profitable review on this occasion. It will, however, help to clear up the situation if we point out that the actual progress of mathematics in the nineteenth century has rendered two widely popular views as to the nature of mathematical knowledge definitely untenable. These views are perhaps best represented by Mill and Schopenhauer, respectively. The first maintains that mathematical knowledge is empirical, inductive, and, therefore, essentially inexact. The second admits it to be deductive but maintains that it is incapable of genuinely extending our knowledge.

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[1] Read at the Princeton meeting of the American Philosophical Association, December 28, 1910.

Mill's position seems to follow from his nominalism. Assuming that only particular physical objects are real, it follows that if mathematics is to be true knowledge it must be of such actual objects as physical squares, physical circles, etc. And as nature does not contain any perfectly straight lines or perfect circles, and as we can not even imagine such entities,[3] mathematics can be only a greater or less approximation to (an otherwise essentially unknowable) physical truth. This would logically lead to the startling result that the more accurate our mathematical methods, the more inaccurate our knowledge of geometric objects. Now whatever plausibility such views may have received from the methods of the infinitesimal calculus prevailing in Mill's day, this can no longer be maintained in view of the increased rigor now prevailing in all branches of mathematics. Mill's contention that even arithmetical truths are contingent, and that in some other universe $2 + 2$ might not equal 4, can now be shown to be utterly devoid of any meaning.[4]

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[2] Kempe's theory was developed in a series of papers in the *Proceedings and Transactions of the Royal Society* (1885 and 1887), but put in less technical form in *Nature,* Vol. 43 (1891), p. 156 f.

[3] Even pure mathematicians like Cayley have remarked that if we had no idea of perfect straightness we never could pass judgment on any particular line and say, This is not perfectly straight. See Cayley's "Presidential Address before the British Association"; *Works,* Vol. XI., p. 432–3.

[4] For the slipshod character of Mill's mathematical logic see Professor W. R. Smith in *Proceedings Edinburgh Royal Society,* Vol. VI., p. 477.

Schopenhauer starts from the assumption that "reason can not actually extend our knowledge, but can only give it another form" ("World as Will," etc., Ch. I., p. 18).

This principle, of course, is shared by many others. Schopenhauer, however, has the courage to carry it to its logical conclusion, viz., that "the differential calculus does not really extend our knowledge of the curve," but that it only transforms what we already know by intuition. Now no one who has ever used the methods of the differential calculus in the solution of actual problems need have the absurdity of this position pointed out to him. Students of mathematics know of numerous instances where analytic methods enable us to discover not only new properties of curves, but even such as on grounds of intuition would seem utterly impossible. To take only one example. It had always been supposed—and on grounds of intuition nothing seemed more certain—that any element of a continuous curve could be extended to form a tangent. But by purely analytic methods Weierstrass discovered a whole host of continuous functions which have no derivatives at all.

It seems, then, to be established that mathematics is both deductive and productive. The tremendous extension of mathematical knowledge in the nineteenth century has gone together with an ever increasing demand for logical rigor, so that a mathematician like Benjamin Peirce does not hesitate to begin his famous memoir on "Linear Associative Algebra" by defining mathematics as "the science which draws necessary conclusions."[5]

By combining the logical movement in mathematics with the generalizing movement in logic, a school of mathematicians, whom I shall call neo-Leibnitzians, are able to maintain the thesis that the whole of pure mathematics is nothing but a development of symbolic logic.

This position has been vehemently attacked from different quarters, but as far as I know, no one has tried to meet the works of Peano, Pieri, Frege, or Russell on their own ground, or has shown the inadequacy of the laborious chain of reasoning by means of which their thesis is established. The opponents of this thesis are satisfied with firing one volley, the query: How can logical principles, which are purely analytic, be productive of genuine novelty or progress in knowledge? Now the way the question *How can?* is used in philosophy as an argument is really scandalous. Suppose a chemist can not answer the question, How can hydrogen and oxygen possibly combine to form a third something utterly different from either?

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[5] *American Journal of Mathematics*, Vol. IV., p. 97. Similarly Dr. Whitehead, "Mathematics in its widest signification is the development of all types of formal necessary deductive reasoning" ("Universal Algebra," Vol. I., p. vi).

Would that excuse us from recognizing the *fact* that, after all, they do so combine? It would seem that when the fact is established that purely logical principles do produce ever new series of true propositions, the question should be not, How *can* they? but How *do* they do so? Be that as it may, we find the problem of "the new" in mathematics officially handed over by the mathematicians to the philosophers as *the* fundamental problem of a philosophy of mathematics.[6]

Approaching the problem from this point of view, we find three types of explanation for the apparent paradox that mathematical knowledge is both deductive and productive.

1. *The Empiricist Explanation.*—Admitting that the body of mathematical science is deductive, modern empiricism maintains that the whole is founded on axioms which are the result of experience. It would take us beyond the limits of this paper to discuss the interesting and important question as to the nature of mathematical axioms; but it is necessary to point out the prevalent confusion between the axioms of mathematics and the so-called axioms of geometry. The latter are genuine hypotheses, the subjunctive clauses of mathematical arguments. Hence we may suppose them not to be true, and deduce genuine consequences from their falsehood. This, however, is not true of the general axioms of mathematics. So far as we can see, no exact or demonstrable reasoning of any kind can be carried on without at least some of them.

Even if we grant that the axioms of mathematics are hypotheses which we are free to make or not according to the system which we want to derive, this would not explain the magic potency by means of which a small number of apparently simple and innocent propositions can produce what Professor Keyser calls the "continent of mathematical doctrines," a continent so extensive that no man since Gauss has ventured to explore the whole of it. The popular conception of the relation between axioms and the propositions which follow from them seems to go back to St. Thomas Aquinas, who holds that axioms are the germs from which the body of knowledge develops.[7] But in spite of the fact that this view is espoused by such a careful investigator as Frege, it can not be regarded as more than a metaphor which fails to explain anything.

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[6] I refer to the preface to the first volume of the "Encyklopädie der Mathematischen Wissenschaften" (p. xxvi), where the editor, Professor Meyer, says: "Möge die Encyklopädie, die die mathematischen Erfindungen eines Jahrhunderts in historischer Entwickelung vorführt, auch das erkenntnistheoretische Studium der grundlegenden Frage, was in der Mathematik denn eigentlich als 'neu' zu gelten habe, beleben!"

[7] "Præexistunt in nobis quædam semina scientarum," "Quæest. disp. de ver.," 11, 1. Similarly Frege, "Grundlagen der Arithmetik," p. 101.

The suggestion that mathematics is actually developed by induction and that the deductive arrangement is simply a reconstruction, would not, even if it were true, obviate the difficulty. For reconstruction, whatever its nature may be, is obviously a construction of some sort, and we are still left with the difficulty of how a deductive procedure can be in any way productive.

2. *The Kantian Explanation.*—The question we are asking is precisely the question which Kant faced when he insisted that mathematical judgments are synthetic *a priori.* Kant defended the synthetic character of mathematical judgments against both Hume and Wolff, and tried to save their *a priori* character by deriving them from the *a priori* intuitions of time and space. Being debarred by various metaphysical considerations from entertaining the possibility of a purely intellectual intuition, these *a priori* intuitions could for Kant relate only to the matter of sensibility. It is a mooted question whether Kant means that mathematical demonstration, as such, depends on intuition, or whether he means that the axioms or basis from which mathematical demonstration proceeds depends on intuition. Philologically there would seem to be no reason for doubting the first interpretation, but in the interest of Kantian apologetics the second has been brought forward.[8] Neither interpretation, however, makes Kant's position really tenable. The progress of mathematical logic and the complete triumph of the movement known as "the arithmetization of mathematics," have forever discredited the view that mathematical demonstration differs from other forms of demonstrative reasoning by some peculiar dependence on the intuition of time and space. The proposition which Kant uses as an example to show that mathematical judgments are synthetic, viz., $7+5=12$, is, as Couturat has amply demonstrated, a purely analytical proposition,—if by analytic we mean demonstrable by purely syllogistic reasoning.[9]

Similarly has the rise of non-Euclidean geometry rendered untenable the view that we have an *a priori* intuition that objective space is Euclidean. It is true that there are not wanting those who argue that non-Euclidean geometry "goes to bear out Kant's theory rather than the reverse." "For he admits [?] that many geometries are possible *logically, i. e.,* geometries working out the consequences of various hypotheses; all that need be asserted is that for us to tell with any certainty which is the geometry of actual space, we must have an *a priori* intuition of space."[10] This argument, however, is based on two assumptions, viz., (1) that we *do* know with absolute certainty that "actual" space is Euclidean, and not, for instance, Riemannian; and (2) that the only ground for such certainty can be an *a priori* intuition. Both of these assumptions are entirely unwarranted.

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[8] L. J. Russel, *Mind,* Vol. XVII., p. 321 sqq. *Cf.* Medicus, "Kant und die nicht-euklidische Geometrie"; Natorp, "Die logischen Grundlagen der exakten Wissenschaften."

[9] "Les principes des mathématiques," pp. 255–6.

[10] L. J. Russel, *Mind,* Vol. XVII., p. 330.

Our physical experience, of course, entirely agrees with, and thus bears out, the Euclidean system; but all empirical results can be interpreted equally well in terms of a non-Euclidean system. Cayley and Klein have shown that the differences between the Euclidean and the two classical non-Euclidean systems are entirely metric and depend on our definition of distance, and Whitehead has given us an elegant proof that if any one of these systems is true, the others must necessarily be true likewise.[11] There is, therefore, as much sense in saying that we have an *a priori* intuition that space is Euclidean, as that we have an *a priori* intuition that space follows the decimal rather than the duodecimal system. The best we can say is that *at present* we find the Euclidean system most convenient. We know nothing, in spite of the high authority of Poincaré, which could prevent us from supposing that the growth of optics, astronomy, or other branches of physics might produce a situation which will make Riemannian geometry the more convenient.[12]

The argument that the Euclidean geometry is the only one which we can represent in intuition has no basis other than the fact that most of us happen to have been taught the Euclidean system in our school days. Indeed, if we disregard analytical considerations, and judge the matter from the point of view of what is possible in intuition, the Lobatchevskian geometry would be the only one that would commend itself.[13] It is interesting to note, in passing, that the professed neo-Kantians in mathematics (Hermann Cohen, Natorp, etc.) have really been forced to give up the whole doctrine of intuition as the productive agency in mathematics.

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[11] "Universal Algebra," art. 262. Cayley's views are put in popular form in his "Presidential Address before the British Association," but are mathematically developed in his "Sixth Memoir on Quantics." *Cf.* Klein, "Die nicht-euklidische Geometrie."

[12] *Cf.* Bôcher's address in *International Congress of Arts and Sciences*, Vol. I., p. 464.

[13] Thus if through a point O, outside of a line XY, a line AB be revolved, it will generally intersect the line XY in some point to the right or left of P, the foot of the perpendicular. Now so long as we are restricted to finite spaces, there is always an angle between the last position of AB in which it intersects XY on the right and the first position $A'B'$ in which it intersects XY on the left of P. This angle BOB' decreases as our space increases; but for any finite space, no matter how large, there always is such an angle. The Lobatchevskian geometry simply asserts that there is such an angle—no matter how small—for the whole of space.

3. *Poincaré's Explanation.*—A more modern attempt to explain the fruitfulness of mathematical reasoning is that of Poincaré, who finds it all due to the principle of mathematical induction. This principle of mathematical induction is undoubtedly of wide application, though there are many regions even in arithmetic where it is difficult to see its application, *e. g.*, the science of prime numbers, a science dealing entirely with non-recurring individuals. But the important thing to observe is that this principle of mathematical induction is entirely different from the induction that prevails in the physical sciences.

What, may we ask, is the essence of induction as practised in the physical sciences? Clearly it is, that in the presence of the empirical observation that all x's are y's, we find an hypothesis A is B, such that, if A is B, it follows that all x's are y's. The Mendelian hypothesis as to heredity will illustrate this as well as any other hypothesis. Now it is well known that in the physical sciences the known facts are never sufficient to establish the hypothesis, in accordance with the simple logical principle that from the fact that, if A is B, it follows that all x's are y's, we can not reason that if all x's are y's, A is B. Physical experience, then, can never prove our hypothesis. It can, however, by turning up some x that is not a y, disprove our hypothesis. Now this is not the case in mathematics, for mathematical reasoning is frequently reversible, *e. g.*, if Euclid's postulate be true, then the sum of all the angles of a triangle is two right angles, but we can also reason that if it is established that the sum of the angles of a triangle is two right angles, then Euclid's postulate is true. But in any case it is absolutely certain that if a proposition is established by mathematical induction, it will never be disproved, *i. e.*, if a *general* proposition is true of $n + 1$ whenever it is true of n, and also of 1, then no possible number can arise of which this proposition is not true, for the principle of mathematical induction is used in defining all finite integers. Whether, therefore, we agree with Russell and call the principle of mathematical induction a definition, or concede to Poincaré that it is a special axiom, a synthetic proposition *a priori*, the fact remains that reasoning from it is a purely deductive procedure.[14]

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[14] It is of course true that many theorems in mathematics are *discovered* inductively, but they are never admitted as mathematically valid unless deductively demonstrated. Thus for over two thousand years before Lindeman in 1880 definitely proved π to be transcendental, mathematicians were unable to derive a rational value for π. Some took the pains to show that π could not be expressed in decimals of over 700 places. Yet this latter fact was never held to be mathematically sufficient to establish the transcendental character of π. So, also, it was nearly 100 years before Fermat's theorem that $2^{2^x} + 1$ is always prime was shown to be false (through Euler's discovery of 4,294,967,297 as

There are two assumptions common to all those who deny the possibility of obtaining genuine knowledge by deduction, viz.: (1) that all deductive reasoning is based on the syllogism and can, therefore, give us nothing but a bare series of identities or tautologies, and (2) that experience is the sole source of truth.

That demonstrative reasoning is not necessarily syllogistic is a thesis which students of logic know to have been successfully maintained against the Aristotelians since the days of Lorenzo Valla, Guelincx, and the Port Royal Logicians. More particularly, that geometric demonstration is not necessarily syllogistic was shown with great force by Cournot (in his "Essai sur les fondements de nos connaissances," T. II., Ch. 16). Whatever objections we may have to modern symbolic logic or logistics, no one can deny that it has established beyond doubt that the syllogism is only one of a number of the principles of demonstrative reasoning; and, moreover, that even the syllogism itself can not be reduced to the principle of identity, that is, the syllogism is more than a bare assertion that *A* is *A*, but is itself a synthetic process.

This brings us to the second point, viz., the doctrine that experience is the sole ground of truth. The term experience as generally used is one of those terrible monsters that, having swallowed up everything, has ceased to have any definite form or outline. As used, however, by the philosophers we are discussing, Kant, Poincaré, and the empiricists, experience obviously means something other than mathematical reasoning,[15] and in this sense the doctrine is demonstrably false. Mathematical reasoning gives us truths if anything does.

Take the standard and much-abused syllogism, Socrates is a man, all men are mortal, therefore Socrates is a mortal. This is usually employed as an example to prove that the syllogism can not give us new truth. For, it is argued, how could I ever know *all* men to be mortal unless I previously knew Socrates to be mortal? Hence it is concluded that the syllogism is simply a case of *petitio principii.* It is to be observed, however, that this argument assumes that a universal proposition can never be established except by a previous

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divisible by 641), and yet at no time during the interval between 1637 and 1732 was this theorem properly a part of mathematical science—simply because no deductive derivation of it had been offered. The truth seems to be that induction is simply more or less systematic guesswork. Mill's well-known inductive methods serve only to eliminate various hypotheses, and thus help us to hit upon the right one; but there can not, in the nature of the case, be any such thing as an inductive proof.

[15] Thus Kant explicitly denies that thinking is an experience. See his profound but, because of its brevity, much-neglected essay, "Ist es ein Erfahrung das wir denken?"

serial knowledge of all possible instances. This, of course, would render all knowledge of universal propositions forever impossible. Yet one can not consistently deny the possibility of knowing universal propositions, for the assertion that no universal propositions are knowable would itself be a universal proposition. It would seem, then, that it is logically impossible for *all* knowledge to be the result of induction (*i. e.,* of a procedure from particulars to universals). Some knowledge, then, must be the result of deduction.

To the question, then, How can reason give us knowledge? we might answer simply that it does, and that there is no valid reason to suppose that it can not do so. We may, perhaps, indicate our answer more positively by saying that reason gives us knowledge in the same way in which ordinary sense perception does so, by presenting an object to us. A series of syllogisms involves a series of perceptions on our part. The *therefore* of each syllogism expresses a perceived fact, viz., that the third proposition or conclusion follows from or is implied by the two premises.[16] This is just as much an ultimate fact as that a particular object looks red or tastes bitter. If my neighbor can not see the redness of apple, or taste its bitterness, there can be no further argument on the matter between us. Neither can there be any further argument between us if he can not see that the propositions, Socrates is a man, and, All men are mortal, imply, Socrates is mortal.

It can not be too rigidly insisted that this fact of logical sequence or implication is not equivalent to the truth of the two premises; for the premises might be false, yet the fact of implication, viz., that they do imply the conclusions, would still be true. Thus most respectable people always have believed that some men, like Elijah and other Biblical heroes, are not mortal; and Semitic scholars at least would not be surprised if some one were to prove that Socrates was not a man but a solar myth; but we can not allow any member of the intellectual kingdom to question the fact that, granted the two premises, the conclusion must follow.

It is to be observed that this fact of implication is just as objective as the facts asserted by the two premises. For it is because of the relation between the two premises, and not from the reasoner's arbitrary fiat, that the conclusion follows. The assertion that propositions are mental would not help us any, for it is obviously not true that any two propositions will imply a definite third proposition. It is because of what the propositions assert that the conclusion follows.

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[16] I use the word *implication,* in this discussion, in its wide and popular sense, and not in the narrower technical sense employed by Russell.

On the other hand, it is equally true that in the apprehension of this simple fact we do not use our sense organs in the same way as we do in the apprehension of physical entities. Every one knows, or ought to know, that if we take the three numbers 105, 252, and 273, the sum of the squares of the first two will equal the square of the third; or that if in a Euclidean space I construct a triangle with its sides respectively 105, 252, and 273 units of any kind, the angle opposite the last side will be a right angle. It is doubtful whether any one ever constructed just such a triangle, and if any one ever did, it would have no bearing on the question. A simple process of reasoning assures us of the truth of this fact in a way in which no amount of physical observation and experiment ever could.

It may be asserted that reasoning itself is a kind of experience, and with this I have no quarrel, provided we recognize it as a kind of experience different from sense experience. The important thing is to beware of the utterly unfounded and pestiferous dogma that reality can be apprehended only through sense perception, a dogma as gratuitous as would be the assumption that all reality is audible, or that all reality is odorous. *The fundamental paralogism of all modern empiricistic systems is the argument that because I can not by pure reason apprehend particular sensible existents, therefore all reality is sensational.* I can not through my sense of sight apprehend the hardness or softness of things, but this does not prove that all reality is tactile, or that colors and other visible realities are non-existent. Through the senses we apprehend one kind of reality and through reason another. Perhaps if we had other organs we might apprehend other kinds or aspects of reality. Human experience, however, functions with these two, and the effort to eliminate from it either the sensible or the intelligible reality is no less ridiculous than the effort to make an army march exclusively with the left or exclusively with the right foot.

The tendency, then, of the newer philosophy of mathematics is to explain the fruitful character of pure mathematics by pointing out that implications are a species of facts as objective or independent of our individual psychology as any of the facts of physics, and that demonstrative reasoning consists of a series of intellectual intuitions or apprehensions of these facts. This also explains the fact that creative genius seems to be necessary to bring about a substantial widening of any field of mathematics. The creative mathematician is the one who has a genius for dealing with our particular species of facts, the ability to grasp long chains of reasonings in their entirety, and the intellectual imagination to anticipate results which less gifted minds reach only by painfully treading every step.

If this view has no other merit, it can certainly claim the one that it enables us to avoid that wilderness of controversy, Are mathematical judgments synthetic or analytic?

II. How is Applied Mathematics Possible?

The second fundamental problem of any philosophy of mathematics is to explain how mathematical reasoning can give us knowledge of the external world—in other words, how applied mathematics is possible. Here again we must not let difficulties of explanation blind us to the certainty of the fact. On any theory we must admit that I can shut myself in my room and, by reasoning from the known density of iron, predict the exact number of seconds it will take for a sound to be transmitted through an iron bar or wire of any length; or, by a few observations on Polaris, tell exactly where and when the sun will rise in a particular locality.

On the view that reasoning is merely a mental process, controlled simply by psychologic laws, the question arises, Why does nature obey the results obtained by following these laws of thought?

The three answers which have prevailed in this field are empiricism, transcendental idealism, and pragmatism.

1. The answer of empiricism, so far as men like Mach and Kirchoff meet this question, is that an applied mathematical science like mechanics is simply a convenient description of phenomena, and the process of reasoning, through equations and the like, simply gives me back what I put into it in the shape of the results of observation and experiment. But this account, strictly adhered to, would make mathematical reasoning useless as a method of extending our knowledge of nature.

2. It is one of the assets of transcendental idealism that it faces this question squarely and gives a definite answer. My reasoning, it says, is able to anticipate nature because nature, after all, is the product of reason—not, indeed, of my individual reason, which is subject to error, but of the absolute reason. Now the nature of the absolute reason or will, and its relation to the empirical self or will, are questions which take us beyond the limits of a philosophy of mathematics. The mathematician must be content with entering a warning that on any metaphysics the laws of mathematics can be no more subject to our empirical will than are physical laws, that the laws of convergent series are no more constituted by our mind than the laws of moving bodies.

3. In speaking of the answer of pragmatism to our present problem, I refer to the view of Poincaré. The distinctive feature of this view is the large rôle given to hypotheses which are merely conventions, and the suggestion that mathematical doctrines are conventional tools for organizing the facts of experience. Mathematical

reasoning holds true of the external physical world because that mode of reasoning which enables us to deal most conveniently with physics has been selected by the process of evolution from the different forms of thinking potentially existing in our minds. The legitimate question, then, according to Poincaré, is not whether Euclidean geometry is true, but whether it is convenient.

In spite of Poincaré's marvelous power of lucidity, the metaphysical bases of this doctrine are essentially obscure. He seems to maintain an agnostic realism—insisting that we can not know the nature of things themselves, but are limited to a knowledge of their relations. He does not explain to us, however, how we can know the relations between things by any principle like mathematical induction, which is based on the mind's ability to intuit its own power.

The answer that seems to me to be suggested by the progress of mathematics itself, is that the relational structure which is the object of mathematics is just as objective—whatever that may mean—as the physical terms related. The laws of mathematics are the laws according to which objective mathematical realities can be combined.

Infinite confusion has resulted from the fact that the laws of logic have been spoken of as the laws of thought. If the laws of thought mean the laws according to which Jones and Smith and others actually think, in those rare moments of their lives when they do, it is hard to see what such psychologic laws have to do with logic. Jones reads Hertz's "Mechanics" and finds it dull and unintelligible. Smith reads it and is charmed to find that he comprehends it in a flash. Neither of these facts, though descriptive of the character of Jones's and Smith's thoughts, has anything to do with the logic of Hertz's "Mechanics"—no more than the question of the stimulants or opiates which Hertz must have used to keep his thoughts on his beautiful demonstrations and to overcome the painful consciousness of the disease which was dragging him to an untimely grave. These considerations are obvious enough, and yet the prevailing tendency is to regard logic as a part of psychology, and to view mathematical operations as merely mental.[17]

There is, of course, another sense in which the laws of logic are spoken of as the laws of thought, viz., as the laws according to which we ought to think if we are to apprehend the real world. The astounding fact, however, is that even the three traditional so-called laws of thought which adorn our logical text-books say nothing at all about thought, but rather make affirmation of existence: whatever is, is; nothing can both be and not be; everything must either be or not be. Would it not be better to call these the invariant laws of being or existence?[18]

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[17] Thus Lipps ("Grundzüge der Logik," p. 2) argues that logic is part of psychology "as surely as knowledge occurs only in the psyche." One might as well argue that astronomy is part of physiology "as surely as vision occurs only in or through the eye."

Instead, therefore, of assuming an alogical nature which somehow or other obeys laws in somebody's mind, would it not be simpler to start from the observed fact that the laws of logic and mathematics do hold of nature, and proceed to inquire what are the other characteristics of nature which follow from or are connected with this fact? Such a procedure involves what the critical philosophy calls dogmatism, viz., the assumption that we do have knowledge. But readers of the history of philosophy know that two such widely different thinkers as Fries and Hegel were each able to point out that the critical philosophy itself is not free from that assumption.

This is not the place to develop the metaphysic that results from such a system, but it is obvious that a view that regards the world of nature as composed of real terms and real relations between them can be neither a chopped up atomism nor a dull monism.

The assumption that numbers and mathematical or logical laws are mental is due to the even more wide-spread notion that only particular sensible entities exist in nature, and that relations, abstractions, or universals can not have any such objective existence—hence they are given a shadowy existence in the mind. But this is a shabby subterfuge: for these numbers or relations are also *numbers and relations of things,* and any assertion with regard to these abstractions is either true or not. Now truth, whatever it is, is not a quality which inheres in a proposition simply because it is mental, but a proposition is true because of factors other than the fact that I now think this proposition. If, therefore, abstractions had no existence except in the mind making them, no assertion into which they entered could possibly be true—except the assertion that I now think such and such a proposition.

The vulgar prejudice against the reality of universals is really due to the fact that we can not point to them and say: *here* they are —that is, they can not be localized in space. But for that matter, neither can our civil rights, our debts, or philosophic misunderstandings and errors; and yet no one has seriously doubted the real existence of the latter. The truth seems to be *that there are different kinds or modes of existence.* But on pragmatic grounds, at any rate, there seems to be no reason why ratios, percentages, or velocities should be considered any less real than the bed-posts or tables which are held up to us by ungenerous brethren as the only genuine types of existence.

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[18] This point is vigorously made by Professor Woodbridge in his paper on "The Field of Logic," *International Congress of Arts and Sciences,* Vol. I.

Let me add a note of confession which perhaps others also might make if they had as little philosophic pride. For several years I have been trying to find out the exact issue between recent idealism and realism, but without avail. Each side seems to me to claim that the other side stands for something which the other side vehemently denies. Each side succeeds in cornering the other when the other isn't there. When, however, I reread my Plato, I find that the differences between him and Protagoras are still the deepest differences which divide philosophers. According to Protagoras[19] we get all of our knowledge through the senses, and thus know things only as they affect us; hence we can not assert the reality of universals, and, therefore, no objective science. The significance of Plato in the history of philosophy is his attempt to disprove every one of these Protagorean propositions; and to my crude mind the fundamental difference among philosophers in our own day is still the question, Plato or Protagoras?

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BRIEF STUDIES IN REALISM. II

EPISTEMOLOGICAL REALISM: THE ALLEGED UBIQUITY OF THE KNOWLEDGE RELATION

AT the close of my previous paper[1] I pointed out that if perception be treated as a case of knowledge, knowledge of every form and kind must be treated as a case of a presentation to a knower. The alleged discipline of epistemology is then inevitable. In common usage, the term knowledge tends to be employed eulogistically; its meaning approaches the connotation of the term science. More loosely, it is used, of course, to designate all beliefs and propositions that are held with assurance, especially with the implication that the assurance is reasonable, or grounded. In its practical sense, it is used as the equivalent of "knowing *how*," of skill or ability involving such acquaintance with things and persons as enables one to anticipate how they behave under certain conditions and to take steps accordingly. Such usages of the term are all differential; they all involve definite contrasts—whether with ungrounded conviction, or with doubt and mere guesswork, or with the inexpertness that accompanies lack of familiarity. In its epistemological use, the term

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[19] I refer to the Platonic Protagoras and not to the historical one. The latter may, for all I know, have been what Gompertz represents him to have been.

[1] This JOURNAL, Vol. VIII., No. 15, p. 393.